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Abstract

The L_∞-estimates of the second derivatives for solutions of the parabolic free boundary problem with two phases

$$\Delta u - \partial_t u = \lambda^+ \chi_{\{u>0\}} - \lambda^- \chi_{\{u<0\}} \text{ in } B_1^+ \times]-1,0], \quad \lambda^\pm \geqslant 0,\ \lambda^+ + \lambda^- > 0,$$

satisfying the non-zero Dirichlet condition on $\Pi_1 := \{(x,t) : |x| \leqslant 1, x_1 = 0, -1 < t \leqslant 0\}$, are proved.

1 Introduction.

In this paper, the optimal regularity for solutions of a parabolic two-phase problem satisfying the non-homogeneous Dirichlet data is proved. Mathematically the problem is formulated as follows.

Let a function u solve the problem:

$$H[u] = \lambda^+ \chi_{\{u>0\}} - \lambda^- \chi_{\{u<0\}} \quad \text{a.e. in} \quad Q_1^+ = B_1^+ \times] - 1, 0], \tag{1}$$

$$u = \varphi \quad \text{on} \quad \Pi_1 := \{(x,t) : |x| \leqslant 1, x_1 = 0, -1 < t \leqslant 0\}, \tag{2}$$

where $H[u] = \Delta u - \partial_t u$ is the heat operator, $\lambda^\pm$ are non-negative constants, $\lambda^+ + \lambda^- > 0$, χ_E is the characteristic function of the set E, $B_1^+ = \{x : |x| < 1, x_1 > 0\}$, and Eq. (1) is satisfied in the sense of distributions.

The local estimates of the the derivatives $\partial_t u$ and $D^2 u$ was proved in [SUW07]. The case $\varphi = 0$ was considered in [Ura07] and the corresponding estimates up to Π_1 were obtained there. We observe that the case of general Dirichlet data cannot be reduce to the case $\varphi = 0$.

We suppose that a given function φ depends only on space variables and satisfies the following conditions:

$$D^3 \varphi \in L_\infty(\Pi_1), \tag{3}$$

$$\exists L > 0 \text{ such that } |D'\varphi(x)| \leqslant L|\varphi(x)|^{2/3} \quad \forall (x,t) \in \Pi_1. \tag{4}$$

We suppose also that $\sup\limits_{Q_1^+} |u| \leqslant M$ with $M \geqslant 1$. Together with (3) it provides for any $\delta \in (0,1)$ the following estimates for u:

$$\|\partial_t u\|_{q, Q_{1-\delta}^+} + \|D^2 u\|_{q, Q_{1-\delta}^+} \leqslant N_1(q, M, \delta, \varphi), \qquad \forall q < \infty, \tag{5}$$

$$\sup_{Q_{1-\delta}^+} |Du| \leqslant N_2(M, \delta, \varphi). \tag{6}$$

$$\frac{|Du(x,t) - Du(y,t^*)|}{|x-y|^\alpha + |t-t^*|^{\alpha/2}} \leqslant N_3(\alpha, M, \delta, \varphi), \quad \forall \alpha \in (0,1). \tag{7}$$

For the corresponding elliptic two-phase problem with Dirichlet data on Π_1 the estimates of the second derivatives of solutions up to Π_1 were obtained by authors in [AU06]. Here we extend the results of [AU06] to the parabolic case.

Theorem. *Let u be a solution of the problem(1)-(2) with a function φ satisfying the assumptions (3) and (4). Suppose also that* $\sup\limits_{Q_1^+} |u| \leqslant M$*.*
Then for any $\delta \in (0, 1/4)$ there exists a positive constant C completely defined by n, M, $\lambda^{\pm}$, δ, L, and by the Sobolev's norm of φ such that

$$ess \sup_{Q_{1-\delta}^+} |D^2 u| \leqslant c.$$

1.1 Notation.

Throughout this paper we use the following notation:
$x = (x_1, x_2, \dots, x_n)$ are points in $\mathbb{R}^n$ with the Euclidean norm $|x|$.
$x \cdot y$ denotes the inner product in $\mathbb{R}^n$.
$e_1, \dots, e_n$ is a standard basis in $\mathbb{R}^n$.
$z = (x, t)$ are points in $\mathbb{R}^{n+1}$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^1$;
χ_E denotes the characteristic function of the set $E \subset \mathbb{R}^{n+1}$;
∂E stands for the boundary of the set E;
$v_+ = \max\{v, 0\}$; $\qquad v_- = \max\{-v, 0\}$;
$B_r(x^0)$ denotes the open ball in $\mathbb{R}^n$ with center x^0 and radius r;
$B_r^+(x_0) = B_r(x_0) \cap \{x_1 > 0\}$;
$Q_r(z^0) = Q_r(x^0, t^0) = B_r(x^0) \times]t^0 - r^2, t^0]$;
$Q_r^+(z^0) = Q_r(z^0) \cap \{x_1 > 0\}$.
When omitted, x^0 (or $z^0 = (x^0, t^0)$, respectively) is assumed to be the origin.
We emphasize that in this paper the top of the cylinder $Q_r(z^0)$ is included in the set $Q_r(z^0)$.
$\partial' Q_r(z^0)$ is the parabolic boundary of $Q_r(z^0)$, i.e., $\partial' Q_r(z^0) = \overline{Q_r(z^0)} \setminus Q_r(z^0)$.
$\Pi_r = \{(x, t) : |x| \leqslant r, x_1 = 0, -r^2 < t \leqslant 0\}$;
$\Pi_r(t^0) = \Pi_r \cap \{t = t^0\}$.

D_i denotes the differential operator with respect to x_i; $\partial_t = \frac{\partial}{\partial t}$;
$D = (D_1, D') = (D_1, D_2, \dots, D_n)$ denotes the spatial gradient;
$D^2 u = D(Du)$ denotes the Hessian of u;
$D^3 u = D(D^2 u)$;
D_ν stands for the operator of differentiation along the direction $\nu \in \mathbb{R}^n$, i.e., $|\nu| = 1$ and

$$D_\nu u = \sum_{i=1}^n \nu_i D_i u.$$

We also emphasize that throughout this paper we will use the symbol ∇ for the whole gradient in the space $\mathbb{R}^n_x \times \mathbb{R}_t$, i.e.

$$\nabla u := (Du, \partial_t u), \qquad |\nabla u| := (|Du|^2 + (\partial_t u)^2)^{1/2}.$$

We adopt the convention regarding summation with respect to repeated indices.

$\| \cdot \|_{p,\,E}$ denotes the norm in $L_p(E)$, $1 < p \leqslant \infty$;
$W^{2,1}_p(E)$ is the anisotropic Sobolev space with the norm

$$\|u\|_{W^{2,1}_p(E)} = \|\partial_t u\|_{p,\,E} + \|D(Du)\|_{p,\,E} + \|u\|_{p,\,E};$$

For a $W^{2,1}_q$-function u defined in Q_1^+, $q < \infty$, we introduce the following set:

$$\Lambda(u) = \{(x,t) \in Q_1^+ : u(x,t) = |Du(x,t)| = 0\}.$$

$\mathcal{H}^m$ stands for the m-dimensional Hausdorff measure.

We use letters M, N, and C (with or without indices) to denote various constants. To indicate that, say, C depends on some parameters, we list them in the parentheses: $C(\dots)$. We will write $C(\varphi)$ to indicate that C is defined by the Sobolev-norms of φ.

1.2 Useful facts

For the reader's convenience and for the future references we recall and explain some facts:

Fact 1. *Each solution of Equation (1) satisfies* $\partial_t u \in L_{\infty,loc}(Q_1^+ \cup \Pi_1)$.
Proof. These statements can be proved analogously to Lemma 3.1 [SUW07], (see also the proofs of Lemma 4.2 [SUW07] and Lemma 3.1 [Ura07]). $\square$

Fact 2. *Let* u *be a solution of Equation (1). Then the set* $\{u = 0\} \cap \{|Du| \neq 0\}$ *is locally in* Q_1^+ *a* C^1*-surface and* $\partial_t u$ *is continuous on that surface. In addition, the unit normal vector to* $\{u = 0\} \cap \{|Du| \neq 0\}$ *directed into* $\{u > 0\}$ *has the form*

$$\gamma(x,t) = \frac{\nabla u(x,t)}{|\nabla u(x,t)|}.$$

Proof. For a proof of this statement we refer the reader to (the proof of) Lemma 7.1 [SUW07]. $\square$

Next statement is a parabolic counterpart of Lemma 2 [AU06].

Fact 3. *Let u be a solution of Equation (1), and let e be a direction in $\mathbb{R}^n$. Then for $(x,t) \in Q_1^+ \setminus \Lambda(u)$ we have*

(i) $H\big[D_e u(x,t)\big] = (\lambda^+ + \lambda^-)\dfrac{D_e u(x,t)}{|\nabla u(x,t)|}\mathcal{H}^{n-1}\lfloor\{u=0,\ |Du| \neq 0\},$

(ii) $H\big[|u(x,t)|\big] = \lambda^+\chi_{\{u>0\}} + \lambda^-\chi_{\{u<0\}} + 2\dfrac{|Du(x,t)|^2}{|\nabla u(x,t)|}\mathcal{H}^{n-1}\lfloor\{u=0,\ |Du| \neq 0\}.$

Proof. Both cases follow from direct computation.

i) Consider an arbitrary direction $e \in \mathbb{R}^n_x$ and a test-function $\eta \in C_0^\infty(Q_1^+ \setminus \Lambda(u))$. Then Eq. (1) together with Facts 1 and 2, and integration by parts provide the following identity

$$
\begin{aligned}
\langle H\big[D_e u\big], \eta\rangle &:= \int D_e u(\partial_t \eta + \Delta\eta)dz = -\int u D_e\left(\partial_t\eta + \Delta\eta\right)dz \\
&= -\int H[u] D_e\eta\, dz = -\lambda^+\int\limits_{\{u>0\}} D_e\eta dz + \lambda^-\int\limits_{\{u<0\}} D_e\eta dz \\
&= \lambda^+\int\limits_{\partial\{u>0\}} \eta\, \cos\big(\widehat{\gamma, e}\big)\, d\mathcal{H}^{n-1} + \lambda^-\int\limits_{\partial\{u<0\}} \eta\, \cos\big(\widehat{\gamma, e}\big)\, d\mathcal{H}^{n-1},
\end{aligned}
$$

where $\gamma = \gamma(x,t)$ is the same vector as in Fact 2.

ii) For any test-function $\eta \in C_0^\infty(Q_1^+ \setminus \Lambda(u))$ the value of distribution $H\big[|u(x,t)|\big]$ on η equals

$$
\langle H\big[|u|\big], \eta\rangle := \int\limits_{\{u>0\}} u(\partial_t\eta + \Delta\eta)dz - \int\limits_{\{u<0\}} u(\partial_t\eta + \Delta\eta)dz.
$$

Integration the last two integrals by parts provides

$$
\begin{aligned}
\langle H\big[|u|\big], \eta\rangle = &\int\limits_{\{u>0\}} (\Delta u - \partial_t u)\eta dz - \int\limits_{\{u<0\}} (\Delta u - \partial_t u)\eta dz \\
&+ 2\int\limits_{\{u=0, |Du|\neq 0\}} (Du \cdot \tilde{\gamma})\, \eta\, d\mathcal{H}^{n-1},
\end{aligned}
$$

where $\tilde{\gamma} = \tilde{\gamma}(x,t)$ ist the projection of $\gamma(x,t)$ onto space $\mathbb{R}^n_x$, i.e., $\tilde{\gamma}(x,t) = \dfrac{Du(x,t)}{|\nabla u(x,t)|}$. Application Eq. (1) to the right-hand side of the above identity finishes the proof. $\square$

2 Lipschitz estimate of the normal derivative at the boundary points

Lemma 1. *Let the assumptions of Theorem hold. Then for arbitrary small $\delta > 0$ there exists constant N_δ such that*

$$|D_\tau u(x,t) - D_\tau \varphi(x',t)| \leqslant N_\delta x_1, \quad \textit{for } (x,t) \in Q^+_{1-\delta}, \quad \tau \perp e_1. \tag{8}$$

The constant N_δ completely defined by δ, n, M, L, $\lambda^\pm$, and by the corresponding Sobolev's norm of φ.

Proof. We fix $\delta \in (0, 1/4)$ and $\tau \in \mathbb{R}^n$, $\tau \perp e_1$.
For arbitrary $t^0 \in (-(1-\delta)^2, 0]$ we consider in the cylinder $Q_{\delta,t^0} = \{(x,t) \in \mathbb{R}^{n+1} : 0 < x_1 < \sqrt{\delta}, |x'| < 1-\delta, t^0 - \delta^2 < t \leqslant t^0\}$, the auxiliary functions

$$v^\pm(x,t) = \pm(D_\tau u(x,t) - D_\tau \varphi(x')) + |u(x,t)| - |\varphi(x')|,$$

and the barrier function

$$w(x,t) = N_4(t^0 - t) + N_5 \left(\frac{x_1}{\sqrt{\delta}} - \frac{x_1^2}{2\delta} \right) + N_6 \left((|x'| - 1 + \delta)_+ \right)^2.$$

Here N_4, N_5 and N_6 are suitable selected positive constants depending only on the parameters of the problem.
It is easy to see that the inequalities

$$v^\pm(x,t^0) \leqslant w(x,t^0) \quad \text{in} \quad Q_{\delta,t^0} \cap \{t = t^0\} \tag{9}$$

together with (6) and arbitrary choice of t^0 imply the desired estimate (8). It remains only to note that inequalities (9) can be established along the same lines as in the proof of Lemma 3 [AU06]. By this reason we omit the detailed verification of (9) here. □

Lemma 2. *Let the assumptions of Theorem hold. Then for arbitrary small $\delta > 0$ and each $t \in (-(1-\delta)^2, 0]$ we have the estimate*

$$|D_1 u(0,x',t) - D_1 u(0,y',t)| \leqslant N_\delta |x' - y'|, \qquad \forall x', y' \in \Pi_{1-\delta}(t), \tag{10}$$

with the same constant N_δ as in Lemma 1.

Proof. If we have the existence of the second derivatives $D'(D_1 u)$ on the surface $\Pi_{1-\delta}$, than Lemma 1 immediately guarantees the boundness of them. However, the derivatives $D'(D_1 u)$ are not defined on $\Pi_{1-\delta}$. By this reason we have to consider instead of u its mollifier with respect to x'-variables u_ε.

It is easy to see that inequality (8) preserves with the same constant N_δ, if we replace in (8) the derivative $D_\tau u$ by $D_\tau u_\varepsilon$ and $D_\tau \varphi$ by $D_\tau \varphi_\varepsilon$, respectively. In other words, from (8) it follows that

$$|D'(D_1 u_\varepsilon)| \leqslant N_\delta \quad \text{in} \quad Q^+_{1-\delta}.$$

The latter inequality means that for $t \in (-(1-\delta)^2, 0]$ and $x', y' \in \Pi_{1-\delta}(t)$ we have, in fact, the estimate

$$|D_1 u_\varepsilon(0, x', t) - D_1 u_\varepsilon(0, y', t)| \leqslant N_\delta |x' - y'|. \tag{11}$$

Now, letting $\varepsilon \to 0$, we get from (11) the desired estimate (10). □

3 Boundary estimates of the second derivatives

Lemma 3. *Let the assumptions of Theorem hold, let an arbitrary $\delta \in (0, 1/4)$ be fixed, and let $z^0 = (x^0, t^0)$ be an arbitrary point on $\Pi_{1-\delta}$. Then for any direction $e \in \mathbb{R}^n$ and a cylinder $Q_r(z^0) \subset Q_{1-\delta}$ we have*

$$\operatorname*{osc}_{Q^+_r(z^0)} D_e u \leqslant C_\delta r, \tag{12}$$

where C_δ depends on the same arguments as the constant N_δ from Lemma 1.

Proof. The proof will be divided into three steps.

$\boxed{\textit{Step } 1.}$ For almost all $t \in (-(1-2\delta)^2, 0)$ the function $u(\cdot, t)$ can be regarded as a solution of an elliptic equation

$$\Delta u(x,t) = F(x) \equiv \lambda^+ \chi_{\{u>0\}} - \lambda^- \chi_{\{u<0\}} + \partial_t u(x,t), \qquad x \in B^+_{1-\delta}.$$

In view of Fact 2 we have $F \in L_\infty(Q^+_{1-\delta})$. Therefore, for a direction $e \in \mathbb{R}^n$ the derivative $D_e u$ satisfies the integral identity

$$\int D(D_e u)\, D\eta dx = \int F\, D_e \eta dx, \qquad \forall \eta \in \overset{\circ}{W}{}^1_2(B^+_{1-\delta}). \tag{13}$$

Setting in the above identity $e = \tau$ with $\tau \perp e_1$ and $\eta = (D_\tau u - D_\tau \varphi)\xi^2$, where ξ is a cut-off function in $B_{2r}(x^0) \subset B_1$, $x_1^{(0)} = 0$, that is equal to 1 in $B_r(x^0)$, we obtain the inequalities

$$\int\limits_{B^+_{2r}(x^0)} |D(D_\tau u(x,t))|^2 \xi^2 dx \leqslant C_\delta r^n, \qquad \tau \perp e_1. \tag{14}$$

Making use of (8) we can easily claim that the constant C_δ in (14) is uniformly bounded with respect to t-variable.
Finally, we find the derivative D_1D_1u from Equation (1) and arrive at the inequality

$$\int\limits_{B_r^+(x^0)} |D^2u(x,t)|^2dx \leqslant C_\delta r^n \tag{15}$$

with uniformly bounded constant C_δ with respect to t-variable.

$\boxed{\textit{Step } 2.}$ We claim that for any direction $e \in \mathbb{R}^n$, and for all $t \in (-(1-\delta)^2, 0]$ and $x \in \Pi_{1-\delta}(t)$ the estimate

$$\operatorname*{osc}_{B_r^+(x)} D_eu(\cdot,t) \leqslant C_\delta r \tag{16}$$

holds true. To prove this, we introduce two auxiliary functions

$$\begin{aligned} K_e(2r,t,x) &:= \sup_{\Pi_1(t)\cap B_{2r}^+(x)} D_eu, \\ k_e(2r,t,x) &:= \inf_{\Pi_1(t)\cap B_{2r}^+(x)} D_eu. \end{aligned}$$

The local estimates for solutions of (13) imply the following inequalities

$$\sup_{B_r^+(x)} D_eu(\cdot,t) \leqslant K_e + N_7\|F\|_{\infty,Q_1^+}r + N_8(n)\sqrt{r^{-n}J_+(t)}, \tag{17}$$

$$\inf_{B_r^+(x)} D_eu(\cdot,t) \geqslant k_e - N_7\|F\|_{\infty,Q_1^+}r - N_8(n)\sqrt{r^{-n}J_-(t)}, \tag{18}$$

with $J_+(t)$ and $J_-(t)$ defined as

$$\begin{aligned} J_+(t) &:= \int\limits_{B_{2r}^+(x)} \left((D_eu(y,t)-K_e)_+\right)^2 dy, \\ J_-(t) &:= \int\limits_{B_{2r}^+(x)} \left((D_eu(y,t)-k_e)_-\right)^2 dy. \end{aligned}$$

Estimating $J_\pm$ with the help of the Poincare inequlity we can conclude that

$$J_\pm \leqslant Cr^2 \int\limits_{B_{2r}^+(x)} (D(D_eu(y,t))^2dy \leqslant Cr^{n+2}, \tag{19}$$

where the second inequality follows from (15).

Combining (11), (17), (18) and (19) we arrive at (16).

$\boxed{\textit{Step } 3.}$ It remains only to verify that $D_1 u$ satisfies on $\Pi_{1-\delta}$ the Hölder condition with respect to t with the exponent $1/2$.
Towards this end, let us consider for $\rho \in [0, \delta)$ the representation

$$\begin{aligned} u(\rho, x', t_1) - u(\rho, x', t_2) &= \int\limits_0^\rho [D_1(s, x', t_1) - D_1(s, x', t_2)]\, ds \\ &= \rho\big[D_1(0, x', t_1) - D_1(0, x', t_2)\big] + \mathcal{I}. \end{aligned} \tag{20}$$

We observe that due to Step 2 $|\mathcal{I}| \leqslant C_\delta \rho^2$. Taking additionally in account the boundedness of the derivatives of $\partial_t u$, we get from (20) the inequality

$$|D_1 u(0, x', t_1) - D_1 u(0, x', t_2)| \leqslant C_\delta \left(\frac{|t_1 - t_2|}{\rho} + \rho \right). \tag{21}$$

It is evident that for $\rho = \sqrt{|t_1 - t_2|}$ the desired Hölder estimate follows immediately from (21). $\square$

Proof of Theorem. Let $\delta \in (0, 1/4)$ and $z^* = (x^*, t^*) \in Q^+_{1-2\delta}$ be fixed, and let $\nu = \frac{Du(z^*)}{|Du(z^*)|}$. Suppose also that e is an arbitrary direction in $\mathbb{R}^n$ if $Du(z^*) = 0$ and $e \perp \nu$ otherwise.
Due to our choice of e we have $D_e u(z^*) = 0$ and, consequently, Lemma 3 provides for $R = x_1^* = \operatorname{dist}\{z^*, \Pi_1\}$ the estimate

$$\sup_{Q_R(z^*)} |D_e u| \leqslant C_\delta R.$$

Now we may apply the result due to L. Caffarelli and C. Kenig [CK98] (see also Lemma 4.2 [Ura07]) to the subcaloric functions $(D_e u)_\pm$ in $Q_R(z^*)$. This leads to the estimate

$$|D(D_e u)(z^*)| \leqslant C_\delta,$$

where C_δ does not depend on R. Since e is an arbitrary direction in $\mathbb{R}^n$ satisfying $e \perp \nu$, the derivative $D_\nu D_\nu u(z^*)$ can be now estimated from Eq. (1). Thus, we have

$$|D^2 u(z^*)| \leqslant C_\delta.$$

$\square$

Remark. It is easy to see that all the arguments hold true if $\varphi = \varphi(x, t)$ and $\partial_t \varphi$ as well as $D(\partial_t \varphi)$ are bounded.

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